11

ERNST'S EQUATION FOR COLLIDING GRAVITATIONAL WAVES

The solutions being considered here for colliding plane waves all have a pair of commuting Killing vectors that are assumed to exist globally. It may therefore be expected that the solutions obtained may be related to known cylindrically symmetric solutions, or to stationary axisymmetric solutions, which similarly have a pair of Killing vectors. Such a relation was first pointed out by Kinnersley (1975), and by Fisher (1980). The exact relation between these solutions has been established more recently by Chandrasekhar and Ferrari (1984), and Chandrasekhar and Xanthopoulos (1985*a*), and exploited by these authors and their colleagues.

In this chapter we will present an analysis of the colliding wave problem using a method that has become familiar in the study of stationary axisymmetric space-times. In this case the field is described in terms of a complex potential function that is referred to as the Ernst potential (see Ernst 1968*a*). This approach leads directly to various methods for generating further exact solutions.

11.1 A derivation of the Ernst equation

First it may be recalled that, of the vacuum field equations (6.22*a*–*f*) considered in previous chapters, (6.22*a*) may immediately be integrated to give

$$e^{-U} = f(u) + g(v), \tag{11.1}$$

and (6.22*d*, *e*) are integrability conditions for the remaining equations. Attention is thus focused on these main equations for the metric functions $V(u,v)$ and $W(u,v)$.

It is now appropriate to consider a different combination of these functions, by putting

$$\chi(u,v) = e^{-V}\mathrm{sech}W, \qquad \omega(u,v) = e^{-V}\tanh W, \tag{11.2}$$

or, inversely

$$e^{V} = (\omega^2+\chi^2)^{-1/2}, \qquad \sinh W = \omega/\chi. \tag{11.3}$$

This modifies the form of the line element (6.20), which now becomes

$$\mathrm{d}s^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - e^{-U}\left(\chi\,\mathrm{d}y^2 + \chi^{-1}(\mathrm{d}x - \omega\mathrm{d}y)^2\right). \tag{11.4}$$

It is also convenient to introduce the complex function

$$Z = \chi + i\omega. \tag{11.5}$$

With this, the line element (11.4) can be written in the form

$$\mathrm{d}s^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - \frac{2e^{-U}}{(Z+\bar{Z})}(\mathrm{d}x + iZ\,\mathrm{d}y)(\mathrm{d}x - i\bar{Z}\,\mathrm{d}y) \tag{11.6}$$

and the two main equations (6.22d, e) can be written as the single complex equation

$$(Z+\bar{Z})(2Z_{uv} - U_uZ_v - U_vZ_u) = 4Z_uZ_v. \tag{11.7}$$

It may be seen that this is in fact Ernst's equation, which can be written in the coordinate-invariant form

$$(Z+\bar{Z})\nabla^2 Z = 2(\nabla Z)^2 \tag{11.8}$$

where $(\nabla\phi)^2 = g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$ is the square of the gradient of an arbitrary scalar field ϕ which, in this case, is a function of the two (null) coordinates only. Similarly, ∇^2 is the 3+1-dimensional Laplacian operator (or the generalized d'Alembertian) given by

$$\nabla^2\phi = (g^{\mu\nu}\phi_{,\nu})_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\mu\nu}\phi_{,\nu}\right)_{,\mu}. \tag{11.9}$$

Using the above notation, the non-zero components of the Weyl tensor given by (6.23) can be written as

$$\begin{aligned}
\Psi_0^\circ &= \frac{|Z|}{Z(Z+\bar{Z})^2}\left[(Z+\bar{Z})(Z_{vv} - U_vZ_v + M_vZ_v) - 2Z_v^2\right] \\
\Psi_2^\circ &= -\frac{U_uU_v}{4} - \frac{\bar{Z}_uZ_v}{(Z+\bar{Z})^2} \\
\Psi_4^\circ &= \frac{|Z|}{Z(Z+\bar{Z})^2}\left[(Z+\bar{Z})(Z_{uu} - U_uZ_u + M_uZ_u) - 2Z_u^2\right].
\end{aligned} \tag{11.10}$$

When considering Ernst's equation, it is frequently found to be useful also to introduce an associated function E, defined by

$$Z = \frac{1+E}{1-E}, \qquad \text{or} \qquad E = \frac{Z-1}{Z+1}. \tag{11.11}$$

With this, the line element (11.4) or (11.6) can be written in the alternative form

$$\begin{aligned} ds^2 &= 2e^{-M}dudv \\ &\quad - \frac{e^{-U}}{1-E\bar{E}}\big[(1-E)dx + i(1+E)dy\big]\big[(1-\bar{E})dx - i(1+\bar{E})dy\big] \end{aligned} \tag{11.12}$$

and the main equations (6.22d, e), or alternatively (11.7), become

$$(1-E\bar{E})(2E_{uv} - U_uE_v - U_vE_u) = -4\bar{E}E_uE_v. \tag{11.13}$$

This is the alternative form of the Ernst equation and may be rewritten in the coordinate-invariant form

$$(1-E\bar{E})\nabla^2E = -2\bar{E}(\nabla E)^2. \tag{11.14}$$

Using this function, the non-zero components of the Weyl tensor (11.10) can be written as

$$\begin{aligned} \Psi_0^{\circ} &= \sqrt{\frac{1-\bar{E}^2}{1-E^2}}\frac{1}{(1-E\bar{E})^2}\left((1-E\bar{E})(E_{vv} - U_vE_v + M_vE_v) + 2\bar{E}E_v^2\right) \\ \Psi_2^{\circ} &= -\frac{U_uU_v}{4} - \frac{\bar{E}_uE_v}{(1-E\bar{E})} \\ \Psi_4^{\circ} &= \sqrt{\frac{1-\bar{E}^2}{1-E^2}}\frac{1}{(1-E\bar{E})^2}\left((1-E\bar{E})(E_{uu} - U_uE_u + M_uE_u) + 2\bar{E}E_u^2\right). \end{aligned} \tag{11.15}$$

The problem now involves finding appropriate solutions of (11.7) or (11.13). These equations, however, contain the derivatives of U, and hence they depend on the arbitrary functions $f(u)$ and $g(v)$ that are specified by the incoming waves. This apparently explicit dependence on initial conditions may in fact be removed by adapting the coordinate system of Chandrasekhar and Ferrari (1984). Here we again use (10.9–12) and put

$$f(u) = \tfrac{1}{2}\cos(\psi+\lambda), \qquad g(v) = \tfrac{1}{2}\cos(\psi-\lambda) \tag{11.16}$$

where ψ and λ are considered as time-like and space-like coordinates, which may then be rescaled by putting

$$t = \sin\psi, \qquad z = \sin\lambda. \tag{11.17}$$

In this coordinate system, equation (11.7) takes the more familiar explicit form of Ernst's equation:

$$\begin{aligned}(Z+\bar{Z})\Big(\big((1-t^2)Z_t\big)_{,t}-\big((1-z^2)Z_z\big)_{,z}\Big)&\\ =2\left((1-t^2)Z_t^2-(1-z^2)Z_z^2\right)&\end{aligned}\tag{11.18}$$

and (11.13) similarly becomes

$$\begin{aligned}(1-E\bar{E})\Big(\big((1-t^2)E_t\big)_{,t}-\big((1-z^2)E_z\big)_{,z}\Big)&\\ =-2\bar{E}\left((1-t^2)E_t^2-(1-z^2)E_z^2\right).&\end{aligned}\tag{11.19}$$

The intermediate steps in the derivation of these equations may be deduced from (16.6) and (16.11). See also (12.30–31).

It may be noticed that, in this case, the solution of these equations immediately determines some of the metric functions. This is in marked contrast to their application in stationary axisymmetric space-times, where the Ernst equation only determines potentials for the fields.

The original metric functions, as considered in previous chapters, are now given by

$$\begin{aligned}e^{2V}&=(Z\bar{Z})^{-1}=\frac{(1-E)(1-\bar{E})}{(1+E)(1+\bar{E})}\\ \sinh W&=-i\frac{(Z-\bar{Z})}{(Z+\bar{Z})}=-i\frac{(E-\bar{E})}{(1-E\bar{E})}.\end{aligned}\tag{11.20}$$

11.2 Boundary conditions

When looking for solutions of the Ernst equation for stationary axisymmetric space-times, it is appropriate to require that solutions be asymptotically flat. However, for colliding plane waves very different boundary conditions apply.

For colliding plane waves it is necessary to choose Z or E, and hence V and W, such that the solution of (6.22b, c, f) for M is continuous across the boundaries of region IV. For vacuum solutions, it is appropriate to use (7.8) and equations (7.9) may then be written in the form

$$\begin{aligned}S_f&=-2(f+g)(1-E\bar{E})^{-2}E_f\bar{E}_f\\ S_g&=-2(f+g)(1-E\bar{E})^{-2}E_g\bar{E}_g.\end{aligned}\tag{11.21}$$

To ensure that the boundary conditions are satisfied, it is then essential that the solution of these equations should include the necessary components (7.10).

In this approach, solutions in the interaction region for $Z(t,z)$ or $E(t,z)$ are related to the functions $V(f,g)$ and $W(f,g)$, and hence S may be obtained as a function of f and g. The boundary conditions described in Chapter 7 may then be considered to place restrictions on the structure of these functions, which characterize the approaching waves.

In practice, these boundary conditions are difficult to apply, basically because the condition that M be continuous is only indirectly applied to the functions V and W, or Z, or E, that feature in the main equations. It is therefore often convenient to apply the boundary condition in the form (7.15) or (7.16). Writing $Z = Z(u,v)$, this becomes

$$\begin{aligned} \lim_{\substack{v\to 0\\ u\to 0}} \left[\frac{2}{n_2(n_2-1)v^{n_2-2}} \frac{Z_v\bar{Z}_v}{(Z+\bar{Z})^2} \right] &= c_2^2 \\ \lim_{\substack{u\to 0\\ v\to 0}} \left[\frac{2}{n_1(n_1-1)u^{n_1-2}} \frac{Z_u\bar{Z}_u}{(Z+\bar{Z})^2} \right] &= c_1^2. \end{aligned} \tag{11.22}$$

This form is particularly convenient when $n_1 = n_2 = 2$, which occurs when impulsive waves are present. The boundary conditions in this limited case only, have been discussed by Ernst, García-Díaz and Hauser (1987*b*).[1]

In terms of the functions f and g, the conditions (11.22) become

$$\begin{aligned} \lim_{\substack{g\to 1/2\\ f\to 1/2}} \left[(\tfrac{1}{2}-g)\frac{Z_g\bar{Z}_g}{(Z+\bar{Z})^2} \right] &= \frac{k_2}{2} \\ \lim_{\substack{f\to 1/2\\ g\to 1/2}} \left[(\tfrac{1}{2}-f)\frac{Z_f\bar{Z}_f}{(Z+\bar{Z})^2} \right] &= \frac{k_1}{2} \end{aligned} \tag{11.23}$$

where k_1 and k_2 must satisfy the inequalities (7.13).

Alternatively, writing $E = E(u,v)$, the boundary conditions require that

$$\begin{aligned} \lim_{\substack{v\to 0\\ u\to 0}} \left[\frac{2}{n_2(n_2-1)v^{n_2-2}} \frac{E_v\bar{E}_v}{(1-E\bar{E})^2} \right] &= c_2^2 \\ \lim_{\substack{u\to 0\\ v\to 0}} \left[\frac{2}{n_1(n_1-1)u^{n_1-2}} \frac{E_u\bar{E}_u}{(1-E\bar{E})^2} \right] &= c_1^2 \end{aligned} \tag{11.24}$$

[1] For an alternative approach to the formulation of the boundary conditions see Hauser and Ernst (1989*a*).

which, in terms of the functions f and g become

$$\begin{aligned}
&\lim_{\substack{g\to 1/2\\ f\to 1/2}} \left[(\tfrac{1}{2}-g)\frac{E_g\bar{E}_g}{(1-E\bar{E})^2} \right] = \frac{k_2}{2} \\
&\lim_{\substack{f\to 1/2\\ g\to 1/2}} \left[(\tfrac{1}{2}-f)\frac{E_f\bar{E}_f}{(1-E\bar{E})^2} \right] = \frac{k_1}{2}.
\end{aligned} \tag{11.25}$$

Finally, it is convenient to express the boundary conditions in terms of the variables t and z. These become, for $Z = Z(t, z)$,

$$\begin{aligned}
&\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(Z_t+Z_z)(\bar{Z}_t+\bar{Z}_z)}{(Z+\bar{Z})^2} \right] = 2k_1 \\
&\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(Z_t-Z_z)(\bar{Z}_t-\bar{Z}_z)}{(Z+\bar{Z})^2} \right] = 2k_2
\end{aligned} \tag{11.26}$$

and for $E = E(t, z)$,

$$\begin{aligned}
&\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(E_t+E_z)(\bar{E}_t+\bar{E}_z)}{(1-E\bar{E})^2} \right] = 2k_1 \\
&\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(E_t-E_z)(\bar{E}_t-\bar{E}_z)}{(1-E\bar{E})^2} \right] = 2k_2
\end{aligned} \tag{11.27}$$

where k_1 and k_2 satisfy (7.13).

11.3 Colinear solutions

In the next chapter, the approach described above will be used to derive new solutions for colliding gravitational waves whose polarization vectors are not aligned. Such solutions essentially have W non-zero, and hence Z and E are complex. However, before moving on to consider such cases, it is appropriate first to review the colinear solutions described in previous chapters. These solutions have Z and E real.

(i) The solution of Khan and Penrose (1971), discussed in Chapter 3 and Section 8.2, which describes the collision of aligned impulsive gravitational waves, is given by

$$Z = \frac{(1+t)}{(1-t)}, \qquad \text{or} \qquad E = t. \tag{11.28}$$

In this case $f = \frac{1}{2} - u^2$ and $g = \frac{1}{2} - v^2$. It may be observed that, for stationary axisymmetric space-times, (11.28) is the Ernst potential which leads to the Schwarzschild solution.

(ii) The Szekeres (1972) class of solutions, described in Chapter 9, is given by

$$Z = \left(\frac{1+t}{1-t}\right)^{(k_1+k_2)/2} \left(\frac{1+z}{1-z}\right)^{(k_1-k_2)/2}. \tag{11.29}$$

In this case $f = \frac{1}{2} - (au)^{n_1} + \dots$ and $g = \frac{1}{2} - (bv)^{n_2} + \dots$, with $k_i^2 = 2(1 - 1/n_i)$ for $i = 1, 2$ and $n_i \geq 2$.

(iii) The 'solution' of Stoyanov (1979) given by (10.24) uses

$$Z = (1 - t^2)^{a/2}(1 - z^2)^{a/2}. \tag{11.30}$$

In Section 10.2 it has been argued that this solution must be considered to be unphysical on its own, but it may be included as a factor in more general solutions. It does not satisfy the boundary conditions (11.26).

(iv) The solution of Ferrari and Ibañez (1987*a*) and Griffiths (1987), which is described in Section 10.3, is characterized by

$$\begin{aligned} Z &= \left(\frac{1+t}{1-t}\right)^{a} e^{b\sigma}, \\ \text{where} \qquad \sigma &= (3z^2 - 1)\left(\tfrac{1}{4}(3t^2 - 1)\log\left(\frac{1+t}{1-t}\right) - \tfrac{3}{2}t\right). \end{aligned} \tag{11.31}$$

In this case $f = \frac{1}{2} - (c_1 u)^n + \dots$ and $g = \frac{1}{2} - (c_2 v)^n + \dots$, where $n \geq 2$ and $(a+b)^2 = 2(1-1/n)$. It may be observed that, when $a = 1$, (11.31) is the Ernst potential which, for stationary axisymmetric solutions, leads to the solution of Erez and Rosen (1959) which describes the external field of a non-rotating body with a quadrupole moment.

(v) The generalized solution of Ferrari and Ibañez (1987*b*), described in Section 10.4, is characterized by

$$Z = \left(\frac{1+t}{1-t}\right)^{k} (1 - t^2)^{a/2}(1 - z^2)^{a/2}. \tag{11.32}$$

This can be seen to include the Stoyanov factor (11.30), and to reduce to a Szekeres solution with $k_1 = k_2 = k$ when $a = 0$. Again $f = \frac{1}{2} - (c_1 u)^n + \dots$ and $g = \frac{1}{2} - (c_2 v)^n + \dots$, where $n \geq 2$ and $k^2 = 2(1-1/n)$. The degenerate cases occur when $k = 1$ and $a = \pm 1$.

(vi) The solution of Tsoubelis and Wang (1989) given by (10.44) has

$$Z = \left(\frac{1+t}{1-t}\right)^{b_1/2} \left(\frac{1+z}{1-z}\right)^{b_2/2} (1-t^2)^{a/2}(1-z^2)^{a/2}. \qquad (11.33)$$

This can be seen to be a generalization of (11.29) and (11.32), and its properties can immediately be deduced.

(vii) The odd order solution of Griffiths (1987) described in Section 10.6 uses the Ernst potential

$$Z = e^{-azQ_1(t)}, \qquad \text{where} \qquad Q_1(t) = \frac{t}{2}\log\left(\frac{1+t}{1-t}\right) - 1. \qquad (11.34)$$

Again $f = \frac{1}{2} - (c_1u)^n + \ldots$ and $g = \frac{1}{2} - (c_2v)^n + \ldots$, where $n \geq 2$ and $a^2 = 8(1 - 1/n)$.

It is not difficult to see how further solutions of this type can be generated.